TOPICS IN SET THEORY: Example Sheet 1 [1]

Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge

Michaelmas 2014-2015 Dr Oren Kolman

1 Fix and state a definition of the cardinality of a set X. Consider the following variants of the Continuum Hypothesis CH.

NI (No Interpolant) There is no infinite set X of real numbers such that the cardinality of X is strictly between those of $\mathbb{N}$ and $\mathbb{R}$.

WOR $< \aleph_2$ (Well-ordered Reals less than order-type $\aleph_2$) Every well-ordering of $\mathbb{R}$ has order-type less than $\aleph_2$.

NOSUR $\aleph_2$ (No surjection onto $\aleph_2$) There is no surjection from $\mathbb{R}$ onto $\aleph_2$.

Using your definition of cardinality, prove that the three variants are equivalent. Indicate any places where you appeal to the axiom of choice (AC).

2 Prove that there exists a subset $S \subseteq \mathbb{R}^2$ that has exactly two points on every line. [S. Mazurkiewicz, C.R. Soc. Sc. et Lettres de Varsovie 7(1914), 382-383.]

3 The Continuum Hypothesis and Sierpiński decompositions of the plane

(i) Prove that $2^{\aleph_0} = \aleph_1$ implies the following assertion SD:

there exist sets A and B such that $\mathbb{R}^2 = A \cup B$ and $A \cap l$ and $B \cap m$ are countable for every horizontal line l and every vertical line m in $\mathbb{R}^2$.
[Hint. Using CH enumerate $\mathbb{R}$ as $\{r_\alpha : \alpha < \omega_1\}$, and consider $A = \{(x, y) : x = r_\alpha, y = r_\beta, \alpha < \beta\}$.]

(ii) Show that SD implies CH.

[Hint. Suppose $2^{\aleph_0} \geq \aleph_2$ and let A and B be a Sierpiński decomposition; let $Z \subseteq \mathbb{R}$ have size $\aleph_1$; for each $y \in \mathbb{R}$ find $z \in Z, (z, y) \notin A$; recall the Pigeonhole Principle.]

(iii) Deduce that CH is equivalent to SD.

Comment. The sets A and B are called a *Sierpiński decomposition.*

4 The Continuum Hypothesis and rainbow colourings

For a set X and cardinal κ, let $[X]^{<\kappa} = \{Y : Y \subseteq X, \mid Y \mid < \kappa\}$.

(i) Prove that $2^{\aleph_0} = \aleph_1$ implies the following assertion Rainbow:

there exists a function $f : \mathbb{R} \to [\mathbb{R}]^{<\aleph_1}$ such that for every uncountable set $X \subseteq \mathbb{R}, \bigcup_{x \in X} f(x) = \mathbb{R}$.

[1] Comments, improvements and corrections will be much appreciated; please send to ok261@cam.ac.uk; rev. 14/12/2014.

(ii) Show that RAINBOW implies CH.

(iii) Deduce that CH is equivalent to RAINBOW.

5 FREILING'S *Axiom of Symmetry* (Sierpiński; Steinhaus; Freiling [1986])

Suppose κ is an infinite cardinal. Recall that $A_{<\kappa}(X)$ is the following assertion:
for every function $f : X \to [X]^{<\kappa}$ there exist x_1 and x_2 such that $x_1 \notin f(x_2)$ and $x_2 \notin f(x_1)$.

(i) Prove (in ZF) that $A_{<2^{\aleph_0}}(\mathbb{R})$ implies that there is no well-ordering of $\mathbb{R}$. [HINT. Reconsider the proof that $\neg CH$ is equivalent to $A_{<\aleph_1}(\mathbb{R})$.]

(ii) Let $A_{null}(\mathbb{R})$ be the assertion: for every function $f : \mathbb{R} \to \{A : A \subseteq \mathbb{R}$ has Lebesgue measure $0\}$ there exist x_1 and x_2 such that $x_1 \notin f(x_2)$ and $x_2 \notin f(x_1)$. Prove (in ZFC) that $A_{null}(\mathbb{R})$ implies that there exists a non-measurable set of cardinality less than $2^{\aleph_0}$ and $\mathbb{R} \neq \bigcup_{\alpha<\omega_1} N_\alpha$ for any family $\{N_\alpha : \alpha < \omega_1\}$ such that every N_α has measure 0.

(iii) Assume $ZFC + 2^{\aleph_0} = \aleph_2$. Prove that $A_{null}(\mathbb{R}) \Leftrightarrow$ (there exists a non-measurable set of cardinality less than $2^{\aleph_0}$ and $\mathbb{R} \neq \bigcup_{\alpha<\omega_1} N_\alpha$ for any family $\{N_\alpha : \alpha < \omega_1\}$ such that every N_α has measure 0).

6 (i) Prove that if X is an uncountable subset of $\mathbb{R}$, then there exists $r \in X$ such that each of the sets $X \cap (-\infty, r)$ and $X \cap (r, \infty)$ is uncountable.

(ii) Prove that if X is an subset of $\mathbb{R}$ of size continuum, then there exists $r \in X$ such that each of the sets $X \cap (-\infty, r)$ and $X \cap (r, \infty)$ is of size continuum.

(iii) Say that a real number r *bisects* the set $X \subseteq \mathbb{R}$ if $\mid X \cap (-\infty, r) \mid = \mid X \cap (r, \infty) \mid$. Prove the following are equivalent:

(a) every uncountable subset $X \subseteq \mathbb{R}$ is bisected by some real r;

(b) $2^{\aleph_0} < \aleph_\omega$.

(iv) Let $BS(\mathbb{R}^2)$ be the assertion: for every uncountable set $S \subseteq \mathbb{R}^2$ there exists a line l such that $\mid S \cap l^+ \mid = \mid S \cap l^- \mid$, where l^+ and l^- are the strict half-planes determined by l. Call such a line a *bisector* of S. Find necessary and sufficient cardinal hypotheses for the provability of $BS(\mathbb{R}^2)$ (if these exist).

(v) Can you suggest any generalizations of $BS(\mathbb{R}^2)$ to e.g. other sorts of line (Sorgenfrey, Suslin, the long line ...), or infinite-dimensional Hilbert or Banach spaces? Do any other lines have the property that the conditions (a) and (b) above are equivalent?

7 The Continuum Hypothesis CH is equivalent to the assertion that there is a family $\{A_\alpha : \alpha < \omega_1\}$ of infinite subsets of ω such that if $X \subseteq \omega$ is infinite, then there is some $\alpha < \omega_1$ such that $A_\alpha \setminus X$ is finite. [F. Rothberger, Fund. Math. 35 (1948), 29-46.]

8 SIERPIŃSKI SETS; LUSIN (MAHLO) SETS

An uncountable set $X \subseteq \mathbb{R}$ is a *Sierpiński set* if $X \cap A$ is countable for every set A of Lebesgue measure 0; an uncountable set $Y \subseteq \mathbb{R}$ is a *Lusin set* if $Y \cap B$ is countable for every set B of first category.

(i) Prove every Sierpiński set is of first category.

(ii) Prove every Lusin set has Lebesgue measure 0.

(iii) CH implies there exists a Sierpiński set.

(iv) CH implies there exists a Lusin set.

(v) Prove that the following are equivalent:

- CH;
- there exists a Sierpiński set and every subset of $\mathbb{R}$ of cardinality less than $2^{\aleph_0}$ has Lebesgue measure 0;
- there exists a Lusin set and every subset of $\mathbb{R}$ of cardinality less than $2^{\aleph_0}$ is of first category.

9 Suppose X is an uncountable complete metric space with a countable base. Prove that X has a subset Y of cardinality $2^{\aleph_0}$ which includes no perfect subset. [Hint. Let $\{C_\alpha : \alpha < 2^{\aleph_0}\}$ list all the closed sets of size at least $2^{\aleph_0}$ (why is this possible?); pick $x_\alpha, y_\alpha \in C_\alpha \setminus \{x_\beta, y_\beta : \beta < \alpha\}, x_\alpha \neq y_\alpha$; consider $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$.]

10 Recall that an infinite abelian group G is *almost free* if and only if every subgroup of G of cardinality less than $|\, G \,|$ is free. Show that CH is equivalent to the assertion that the group $\mathbb{Z}^\omega$ is almost free, where $\mathbb{Z}^\omega$ is the group of integer-valued sequences under component-wise addition. [You may assume without proof the fact that $\mathbb{Z}^\omega$ is $\aleph_1$-free; you should try to prove that $\mathbb{Z}^\omega$ is not $\aleph_2$-free: pick a prime p, consider a subgroup H of size $\aleph_1$, containing the direct sum $\bigoplus_{n\in\omega} \mathbb{Z}$, and consisting of elements whose "tails" are divisible by arbitrarily high powers of p; towards a contradiction, compare the cardinalities of H and H/pH if H were free.]

11 Cardinal invariants/characteristics

Define the quasi-order $\preceq^*$ on the set ${}^\omega\omega = \{f : f \text{ is a function from } \omega \text{ to } \omega\}$ as follows:

$$f \preceq^* g \Leftrightarrow f(n) \leq g(n) \text{ for all but finitely many } n \in \omega .$$

A subset $B \subseteq {}^\omega\omega$ is *unbounded* if B is unbounded in $({}^\omega\omega, \preceq^*)$; a subset $D \subseteq^\omega \omega$ is *dominating* if D is *cofinal* in $({}^\omega\omega, \preceq^*)$ (i.e. $(\forall f \in^\omega \omega)(\exists g \in D)(f \preceq^* g)$); a dominating set D is a *scale* if D is well-ordered by $\preceq^*$. Let $\mathfrak{a} = min\{|\, A \,|: A$ is an infinite maximal almost disjoint family of subsets of $\omega\}$, $\mathfrak{b} = min\{|\, B \,|: B$ is unbounded in $({}^\omega\omega, \preceq^*)\}$, and $\mathfrak{d} = min\{|\, D \,|: D$ is dominating in $({}^\omega\omega, \preceq^*)\}$. The cardinals $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{d}$ are the simplest examples of *cardinal invariants* (or characteristics) of the continuum.

(i) Prove that $\aleph_1 \leq min\{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\} \leq max\{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\} \leq 2^{\aleph_0}$.

(ii) Prove that $\mathfrak{b} \leq \mathfrak{a}$.

(iii) Prove that $\mathfrak{b} \leq \mathfrak{d}$.

Comment. Cardinal invariants of the continuum have proliferated and are intensively studied. They frequently give rise to independence phenomena; ZFC-provable strict inequalities and equations are rare and elusive. An introductory account is given by Eric van Douwen, The integers and topology, in: Kunen, K., Vaughan, J.E., *Handbook of Set-theoretic Topology*, Elsevier Science Publishers, 1984; for an advanced recent survey, see Andreas Blass, Combinatorial characteristics of the Continuum, in: Kanamori, A., Foreman, M., *Handbook of Set Theory*, Springer, Berlin, 2012.

12 Ulam Matrices

(i) Prove there exists a family $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ of sets such that

(1) for every $\alpha < \omega_1$, the set $\omega_1 \setminus \bigcup_{n<\omega} A_{\alpha,n}$ is countable;

(2) if $\alpha \neq \beta < \omega_1$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for all $n < \omega$.

[Hint. For each ordinal $\alpha < \omega_1$, choose a surjection f_α from ω onto α (why is this possible?), and now consider the set $A_{\alpha,n} = \{\xi : f_\xi(n) = \alpha\}$.]

(ii) Determine whether the following assertion is provable: there exists a family $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ of sets such that

(1) for every $\alpha < \omega_1$, the set $\omega_1 \setminus \bigcup_{n<\omega} A_{\alpha,n}$ is finite;

(2) if $\alpha \neq \beta < \omega_1$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for all $n < \omega$.

13 (i) Show that if D is a dense subset of the linear order $(\mathbb{R}, <)$ and $\varphi : D \to D$ is an order-automorphism, then φ has a unique extension to an order-automorphism $\tilde{\varphi}$ of $(\mathbb{R}, <)$.

(ii) Prove that any order-automorphism of $(\mathbb{R}, <)$ is determined by its values on $\mathbb{Q}$. Deduce that there are exactly $2^{\aleph_0}$ order-automorphisms of $(\mathbb{R}, <)$.

(iii) Prove that there exists an uncountable rigid dense subset of the real line which has no non-trivial order-automorphism, i.e. no order-automorphism other than the identity. [B. Dushnik, E.W. Miller, Concerning similarity transformations of linearly ordered sets, Bull. Amer. Math. Soc. 46 (1940), 322–326.]

(iv) Deduce that CH implies the existence of a rigid dense set of reals of size $\aleph_1$.

14 (i) Suppose that $\aleph_1^{\aleph_0} = \aleph_1$. Prove for all non-zero $n < \omega, \aleph_n^{\aleph_0} = \aleph_n$.

(ii) Recall that the beth function on Ord is defined as follows: $\beth_0 = \aleph_0, \beth_{\alpha+1} = 2^{\beth_\alpha}, \beth_\delta = \bigcup_{\alpha<\delta} \beth_\alpha$ for limit ordinals δ. Prove that $\beth_{\omega_1}^{\aleph_0} < \beth_{\omega_1}^{\aleph_1}$.

15 Prove that for every non-zero ordinal $\alpha < \omega_2$, there is a surjection from $\mathbb{R}$ onto α. Now do it without assuming AC.

16 Question.

Can one prove the existence of a rigid dense set D of real numbers of size $\aleph_1$ in ordinary set theory?